Problem 4.7

Using Equation 4.32, find $Y_\ell^\ell(\theta,\phi)$ and $Y_3^2(\theta,\phi)$. (You can take P_3^2 from Table 4.2, but you'll have to work out P_ℓ^ℓ from Equations 4.27 and 4.28.) Check that they satisfy the angular equation (Equation 4.18), for the appropriate values of ℓ and m.

Solution

Equation 4.32 on page 137 is the formula for the spherical harmonics.

$$Y_\ell^m(\theta,\phi) = \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\,e^{im\phi}P_\ell^m(\cos\theta) \tag{4.32}$$

$P_\ell^m(x)$ are the associated Legendre functions, which can written in terms of the Legendre functions $P_\ell(x)$.

$$\begin{aligned} P_\ell^m(x) &= (-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_\ell(x) \\ &= \frac{(-1)^m}{2^\ell \ell!}(1-x^2)^{m/2}\frac{d^{\ell+m}}{dx^{\ell+m}}(x^2-1)^\ell \end{aligned}$$

Find $Y_3^2(\theta,\phi)$ first.

$$\begin{aligned} Y_3^2(\theta,\phi) &= \sqrt{\frac{[2(3)+1]}{4\pi}\frac{(3-2)!}{(3+2)!}}\,e^{2i\phi}P_3^2(\cos\theta) \\ &= \sqrt{\frac{7}{4\pi}\frac{1!}{5!}}\,e^{2i\phi}\left[\frac{(-1)^2}{2^3 3!}(1-x^2)^{2/2}\frac{d^{3+2}}{dx^{3+2}}(x^2-1)^3\right]\Bigg|_{x=\cos\theta} \\ &= \sqrt{\frac{7}{480\pi}}\,e^{2i\phi}\left[\frac{1}{48}(1-x^2)\frac{d^5}{dx^5}(x^6-3x^4+3x^2-1)\right]\Bigg|_{x=\cos\theta} \\ &= \frac{1}{4}\sqrt{\frac{7}{30\pi}}\,e^{2i\phi}\left[\frac{1}{48}(1-x^2)\frac{d^4}{dx^4}(6x^5-12x^3+6x)\right]\Bigg|_{x=\cos\theta} \\ &= \frac{1}{4}\sqrt{\frac{7}{30\pi}}\,e^{2i\phi}\left[\frac{1}{48}(1-x^2)\frac{d^3}{dx^3}(30x^4-36x^2+6)\right]\Bigg|_{x=\cos\theta} \\ &= \frac{1}{4}\sqrt{\frac{7}{30\pi}}\,e^{2i\phi}\left[\frac{1}{48}(1-x^2)\frac{d^2}{dx^2}(120x^3-72x)\right]\Bigg|_{x=\cos\theta} \\ &= \frac{1}{4}\sqrt{\frac{7}{30\pi}}\,e^{2i\phi}\left[\frac{1}{48}(1-x^2)\frac{d}{dx}(360x^2-72)\right]\Bigg|_{x=\cos\theta} \\ &= \frac{1}{4}\sqrt{\frac{7}{30\pi}}\,e^{2i\phi}\left[\frac{1}{48}(1-x^2)(720x)\right]\Bigg|_{x=\cos\theta} \\ &= \frac{15}{4}\sqrt{\frac{7}{30\pi}}\,e^{2i\phi}(1-\cos^2\theta)\cos\theta \end{aligned}$$

Therefore,

$$Y_3^2(\theta,\phi) = \frac{1}{4}\sqrt{\frac{105}{2\pi}}\,e^{2i\phi}\cos\theta\sin^2\theta.$$

Check to see that it satisfies the angular equation (Equation 4.18 on page 134),

$$\sin\theta\,\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y}{\partial\theta}\right) + \frac{\partial^2 Y}{\partial\phi^2} = -\ell(\ell+1)\sin^2\theta\,Y, \tag{4.18}$$

for $\ell = 3$.

$$\begin{aligned}
\sin\theta\,\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y_3^2}{\partial\theta}\right) + \frac{\partial^2 Y_3^2}{\partial\phi^2} &= \sin\theta\,\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{4}\sqrt{\frac{105}{2\pi}}\,e^{2i\phi}\cos\theta\sin^2\theta\right)\right] + \frac{\partial^2}{\partial\phi^2}\left(\frac{1}{4}\sqrt{\frac{105}{2\pi}}\,e^{2i\phi}\cos\theta\sin^2\theta\right) \\
&= \frac{e^{2i\phi}\sin\theta}{4}\sqrt{\frac{105}{2\pi}}\frac{d}{d\theta}\left[\sin\theta\,\frac{d}{d\theta}(\cos\theta\sin^2\theta)\right] + \frac{1}{4}\sqrt{\frac{105}{2\pi}}\,\cos\theta\sin^2\theta\,\frac{d^2}{d\phi^2}(e^{2i\phi}) \\
&= \frac{1}{4}\sqrt{\frac{105}{2\pi}}\left\{e^{2i\phi}\sin\theta\,\frac{d}{d\theta}\left[\sin\theta\,\frac{d}{d\theta}(\cos\theta\sin^2\theta)\right] + \cos\theta\sin^2\theta\,\frac{d^2}{d\phi^2}(e^{2i\phi})\right\} \\
&= \frac{1}{4}\sqrt{\frac{105}{2\pi}}\left\{e^{2i\phi}\sin\theta\,\frac{d}{d\theta}\left[\sin\theta(-\sin^3\theta + 2\cos^2\theta\sin\theta)\right] + \cos\theta\sin^2\theta\,\frac{d}{d\phi}(2ie^{2i\phi})\right\} \\
&= \frac{1}{4}\sqrt{\frac{105}{2\pi}}\left[e^{2i\phi}\sin\theta\,\frac{d}{d\theta}(2\cos^2\theta\sin^2\theta - \sin^4\theta) + \cos\theta\sin^2\theta\,\frac{d}{d\phi}(2ie^{2i\phi})\right] \\
&= \frac{1}{4}\sqrt{\frac{105}{2\pi}}\left[e^{2i\phi}\sin\theta\,\frac{d}{d\theta}[2(1-\sin^2\theta)\sin^2\theta - \sin^4\theta] + \cos\theta\sin^2\theta\,\frac{d}{d\phi}(2ie^{2i\phi})\right] \\
&= \frac{1}{4}\sqrt{\frac{105}{2\pi}}\left[e^{2i\phi}\sin\theta\,\frac{d}{d\theta}(2\sin^2\theta - 3\sin^4\theta) + \cos\theta\sin^2\theta\,\frac{d}{d\phi}(2ie^{2i\phi})\right] \\
&= \frac{1}{4}\sqrt{\frac{105}{2\pi}}\left[e^{2i\phi}\sin\theta\,(4\sin\theta\cos\theta - 12\sin^3\theta\cos\theta) + \cos\theta\sin^2\theta\,(4i^2e^{2i\phi})\right] \\
&= \frac{1}{4}\sqrt{\frac{105}{2\pi}}\left(\cancel{4e^{2i\phi}\cos\theta\sin^2\theta} - 12e^{2i\phi}\cos\theta\sin^4\theta - \cancel{4e^{2i\phi}\cos\theta\sin^2\theta}\right) \\
&= -(12\sin^2\theta)\left(\frac{1}{4}\sqrt{\frac{105}{2\pi}}\,e^{2i\phi}\cos\theta\sin^2\theta\right) \\
&= -3(3+1)\sin^2\theta\,Y_3^2(\theta,\phi)
\end{aligned}$$

Now determine $Y_\ell^\ell(\theta,\phi)$.

$$\begin{aligned} Y_\ell^\ell(\theta,\phi) &= \sqrt{\frac{(2\ell+1)}{4\pi}\frac{(\ell-\ell)!}{(\ell+\ell)!}}\,e^{i\ell\phi}P_\ell^\ell(\cos\theta) \\ &= \sqrt{\frac{2\ell+1}{4\pi}\frac{0!}{(2\ell)!}}\,e^{i\ell\phi}\left[\frac{(-1)^\ell}{2^\ell\ell!}(1-x^2)^{\ell/2}\frac{d^{\ell+\ell}}{dx^{\ell+\ell}}(x^2-1)^\ell\right]\Bigg|_{x=\cos\theta} \\ &= \sqrt{\frac{2\ell+1}{4\pi(2\ell)!}}\,e^{i\ell\phi}\left[\frac{(-1)^\ell}{2^\ell\ell!}(1-x^2)^{\ell/2}\frac{d^{2\ell}}{dx^{2\ell}}(x^2-1)^\ell\right]\Bigg|_{x=\cos\theta} \end{aligned}$$

Take the 2ℓth derivative of $(x^2-1)^\ell$ for several values of ℓ and try to find a pattern.

$$\begin{aligned} \ell=0: &\quad \frac{d^0}{dx^0}(x^2-1)^0 = (x^2-1)^0 = 1 \\ \ell=1: &\quad \frac{d^2}{dx^2}(x^2-1)^1 = \frac{d^2}{dx^2}(x^2-1) = 2 \\ \ell=2: &\quad \frac{d^4}{dx^4}(x^2-1)^2 = \frac{d^4}{dx^4}(x^4-2x^2+1) = 24 \\ \ell=3: &\quad \frac{d^6}{dx^6}(x^2-1)^3 = \frac{d^6}{dx^6}(x^6-3x^4+3x^2-1) = 720 \\ \ell=4: &\quad \frac{d^8}{dx^8}(x^2-1)^4 = \frac{d^8}{dx^8}(x^8-4x^6+6x^4-4x^2+1) = 40\,320 \\ &\quad \vdots \\ &\quad \frac{d^{2\ell}}{dx^{2\ell}}(x^2-1)^\ell = (2\ell)! \end{aligned}$$

Consequently,

$$\begin{aligned} Y_\ell^\ell(\theta,\phi) &= \sqrt{\frac{2\ell+1}{4\pi(2\ell)!}}\,e^{i\ell\phi}\left[\frac{(-1)^\ell}{2^\ell\ell!}(1-x^2)^{\ell/2}(2\ell)!\right]\Bigg|_{x=\cos\theta} \\ &= \frac{(-1)^\ell}{2^\ell\ell!}\sqrt{\frac{(2\ell+1)(2\ell)!}{4\pi}}\,e^{i\ell\phi}(1-\cos^2\theta)^{\ell/2} \\ &= \frac{(-1)^\ell}{2^\ell\ell!}\sqrt{\frac{(2\ell+1)!}{4\pi}}\,e^{i\ell\phi}(\sin^2\theta)^{\ell/2} \\ &= \frac{(-1)^\ell}{2^\ell\ell!}\sqrt{\frac{(2\ell+1)!}{4\pi}}\,e^{i\ell\phi}\sin^\ell\theta. \end{aligned}$$

Check to see that it satisfies the angular equation.

$$\begin{aligned}
\sin\theta\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial Y_\ell^\ell}{\partial\theta}\right)+\frac{\partial^2 Y_\ell^\ell}{\partial\phi^2} &= \sin\theta\frac{\partial}{\partial\theta}\left\{\sin\theta\frac{\partial}{\partial\theta}\left[\frac{(-1)^\ell}{2^\ell\ell!}\sqrt{\frac{(2\ell+1)!}{4\pi}}\,e^{i\ell\phi}\sin^\ell\theta\right]\right\}+\frac{\partial^2}{\partial\phi^2}\left[\frac{(-1)^\ell}{2^\ell\ell!}\sqrt{\frac{(2\ell+1)!}{4\pi}}\,e^{i\ell\phi}\sin^\ell\theta\right]\\
&= \frac{(-1)^\ell}{2^\ell\ell!}\sqrt{\frac{(2\ell+1)!}{4\pi}}\,e^{i\ell\phi}\sin\theta\frac{d}{d\theta}\left[\sin\theta\frac{d}{d\theta}\left(\sin^\ell\theta\right)\right]+\frac{(-1)^\ell}{2^\ell\ell!}\sqrt{\frac{(2\ell+1)!}{4\pi}}\,\sin^\ell\theta\frac{d^2}{d\phi^2}\left(e^{i\ell\phi}\right)\\
&= \frac{(-1)^\ell}{2^\ell\ell!}\sqrt{\frac{(2\ell+1)!}{4\pi}}\,e^{i\ell\phi}\sin\theta\frac{d}{d\theta}\left[\sin\theta\left(\ell\sin^{\ell-1}\theta\cos\theta\right)\right]+\frac{(-1)^\ell}{2^\ell\ell!}\sqrt{\frac{(2\ell+1)!}{4\pi}}\,\sin^\ell\theta\frac{d}{d\phi}\left(i\ell e^{i\ell\phi}\right)\\
&= \frac{(-1)^\ell}{2^\ell\ell!}\sqrt{\frac{(2\ell+1)!}{4\pi}}\,\ell e^{i\ell\phi}\sin\theta\frac{d}{d\theta}\left(\sin^\ell\theta\cos\theta\right)+\frac{(-1)^\ell}{2^\ell\ell!}\sqrt{\frac{(2\ell+1)!}{4\pi}}\,\sin^\ell\theta\frac{d}{d\phi}\left(i\ell e^{i\ell\phi}\right)\\
&= \frac{(-1)^\ell}{2^\ell\ell!}\sqrt{\frac{(2\ell+1)!}{4\pi}}\left[\ell e^{i\ell\phi}\sin\theta\frac{d}{d\theta}\left(\sin^\ell\theta\cos\theta\right)+\sin^\ell\theta\frac{d}{d\phi}\left(i\ell e^{i\ell\phi}\right)\right]\\
&= \frac{(-1)^\ell}{2^\ell\ell!}\sqrt{\frac{(2\ell+1)!}{4\pi}}\left[\ell e^{i\ell\phi}\sin\theta\left(\ell\sin^{\ell-1}\theta\cos^2\theta-\sin^{\ell+1}\theta\right)+\sin^\ell\theta\left(i^2\ell^2e^{i\ell\phi}\right)\right]\\
&= \frac{(-1)^\ell}{2^\ell\ell!}\sqrt{\frac{(2\ell+1)!}{4\pi}}\left(\ell^2e^{i\ell\phi}\sin^\ell\theta\cos^2\theta-\ell e^{i\ell\phi}\sin^{\ell+2}\theta-\ell^2e^{i\ell\phi}\sin^\ell\theta\right)\\
&= \frac{(-1)^\ell}{2^\ell\ell!}\sqrt{\frac{(2\ell+1)!}{4\pi}}\left(\ell\cos^2\theta-\sin^2\theta-\ell\right)\ell e^{i\ell\phi}\sin^\ell\theta\\
&= \frac{(-1)^\ell}{2^\ell\ell!}\sqrt{\frac{(2\ell+1)!}{4\pi}}\left[\ell(1-\sin^2\theta)-\sin^2\theta-\ell\right]\ell e^{i\ell\phi}\sin^\ell\theta\\
&= \frac{(-1)^\ell}{2^\ell\ell!}\sqrt{\frac{(2\ell+1)!}{4\pi}}\left(-\ell\sin^2\theta-\sin^2\theta\right)\ell e^{i\ell\phi}\sin^\ell\theta\\
&= -\ell(\ell+1)\sin^2\theta\left[\frac{(-1)^\ell}{2^\ell\ell!}\sqrt{\frac{(2\ell+1)!}{4\pi}}\,e^{i\ell\phi}\sin^\ell\theta\right]\\
&= -\ell(\ell+1)\sin^2\theta\,Y_\ell^\ell(\theta,\phi)
\end{aligned}$$